**COMP 585** **Spring 2026**

*Probabilistic Toolkit for Learning and Computing* *Instructor: Maryam Aliakbarpour*

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# Lecture 1

## Probability Review

To lay the groundwork for understanding randomized algorithms and their analysis, we'll start with a review of fundamental probability concepts. While we expect students to have some familiarity with these ideas, this lecture serves as a quick refresher.

A discrete probability space is defined by $\Omega = \{w_1, w_2, \dots\}$, which represents the sample space containing elementary outcomes (finite or countable). An event is any subset of $\Omega$. The probability function $P$ maps events to values in $[0, 1]$ such that $\sum_{\omega \in \Omega} P(\omega) = 1$. For any event $E \subseteq \Omega$, $P(E)$ is defined as the sum of probabilities of elementary outcomes in $E$: $P(E) \coloneqq \sum_{\omega \in E} P(\omega)$.

### Discrete Probability Spaces

The probability measure $P$ is defined by assigning a probability $P(w_i) \in [0, 1]$ to each elementary outcome $w_i \in \Omega$, such that:

$$\sum_i P(w_i) = 1.$$

The probability of any event $E \subseteq \Omega$ is given by:

$$P(E) \coloneqq \sum_{\omega \in E} P(\omega).$$

**Example: Rolling a Fair Die.** Consider a single roll of a fair six-sided die. The sample space is

$$\Omega = \{1, 2, 3, 4, 5, 6\},$$

and since the die is fair, each outcome has probability $P(\omega) = 1/6 \quad \text{for all } \omega \in \Omega$. Let $A$ be the event that the outcome is even: $A = \{2, 4, 6\}$. Then, the probability of this event is: $\mathbf{Pr}[A] = 3/6 = 1/2$.

## Continuous Probability Spaces

If we define a probability distribution more formally, a probability space is a triple $(\Omega, \mathcal{F}, P)$, where:

- $\Omega$ is the **sample space**, the set of all possible outcomes,
- $\mathcal{F}$ is a **$\sigma$-algebra** of subsets of $\Omega$, whose elements are called **measurable sets** or **events**,
- $P$ is a **probability measure** defined on $\mathcal{F}$, assigning each event a value in $[0, 1]$ such that $P(\Omega) = 1$, and $P$ is countably additive.

A $\sigma$-algebra $\mathcal{F}$ is a collection of subsets of $\Omega$ that includes the empty set, is closed under complements, and is closed under countable unions (and hence countable intersections). These properties ensure that probabilities are well-defined for limits of events and match our intuition about how events should behave. We will not study the formal construction of $\sigma$-algebras here; instead, we treat them as specifying which events can be meaningfully assigned probabilities, and are therefore called *measurable*. A subset $E \subseteq \Omega$ is called measurable if $E \in \mathcal{F}$.

In the discrete case, the sample space $\Omega = \{w_1, w_2, \dots\}$ is finite or countably infinite. The $\sigma$-algebra $\mathcal{F}$ is typically the power set $2^\Omega$, so every subset is measurable. In the continuous case, the sample space $\Omega$ is uncountable (for example, an interval of real numbers). The $\sigma$-algebra $\mathcal{F}$ is usually taken to be the Borel $\sigma$-algebra or a closely related collection of **Lebesgue-measurable** sets.

In such settings, a probability distribution $P$ assigns probabilities to events in $\mathcal{F}$ such that $P(\Omega) = 1$ and is countably additive. Both discrete and continuous probability spaces fit into the same measure-theoretic framework. The main distinction lies in how the measure $P$ is constructed: as a weighted sum over points in the discrete case, and as an integral with respect to a density in the continuous case. In both settings, the $\sigma$-algebra $\mathcal{F}$ ensures that probabilities are well-defined and behave properly under limits.

### Random variable, expectation, and variance

A random variable takes values based on a probability distribution. We denote $X \sim P$, if $X$ takes value $x$ with probability $P(x)$. If $X \sim P$, its expected value and its variance are defined as follows:

$$\mathbf{E}[X] \coloneqq \sum_{x \in \Omega} P(x) \cdot x \,,$$
$$\mathbf{Var}[X] \coloneqq \mathbf{E}\big[(X - \mathbf{E}[X])^2\big] \,.$$

Some key properties of expectations and variances are as follows:

- Linearity of Expectation:

$$\text{For every pair of random variables } X \text{ and } Y: \quad \mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]\,,$$
$$\text{For all constant scalar } a \in \mathbb{R}: \quad \mathbf{E}[a \cdot X] = a \cdot \mathbf{E}[X]\,.$$

- Alternative form: *integral identity* for expectation, states that for a **non-negative** random variable $X$, we have:

$$\mathbf{E}[X] = \int_0^\infty \mathbf{Pr}[X > t]\,dt\,.$$

- Linearity of variance under independence:

$$\text{For every pair of \textbf{independent} random variables } X \text{ and } Y: \quad \mathbf{Var}[X+Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]\,,$$
$$\text{For all constant scalar } a \in \mathbb{R}: \quad \mathbf{Var}[a \cdot X] = a^2 \cdot \mathbf{Var}[X]\,.$$

- Alternative formula for variance:

$$\mathbf{Var}[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2\,.$$

- Law of total variance: Suppose we have two random variables $X$ and $Y$ that are on the same probability space, and $Y$ has finite variance, then:

$$\mathbf{Var}[Y] = \mathbf{E}_X[\mathbf{Var}[Y|X]] + \mathbf{Var}_X[\mathbf{E}[Y|X]]\,.$$

**Example:** Let $X$ be the random variable equal to the outcome of the die roll. Then

$$\mathbf{E}[X] = \sum_{x=1}^{6} x \cdot \frac{1}{6} = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}.$$

We compute the variance using the alternative formula

$$\mathbf{Var}[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2.$$

First, we compute

$$\mathbf{E}\left[X^2\right] = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \frac{91}{6}.$$

Thus,

$$\mathbf{Var}[X] = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

Hence, the variance of a fair six-sided die is $\mathbf{Var}[X] = \frac{35}{12}$.

### Joint and Conditional Probability

For two events $A$ and $B \subseteq \Omega$, the joint probability, $\mathbf{Pr}[A \cap B]$, is the probability that both $A$ and $B$ happen. Conditional probability, $P_{|A}(B)$ or $\mathbf{Pr}[B|A]$, is the probability that $B$ happens conditioned on that $A$ happens, defined as:

$$\mathbf{Pr}[B|A] = \frac{\mathbf{Pr}[A \cap B]}{\mathbf{Pr}[A]}\,.$$

Bayes' Theorem states that

$$\mathbf{Pr}[A|B] = \frac{\mathbf{Pr}[B|A] \cdot \mathbf{Pr}[A]}{\mathbf{Pr}[B]}\,.$$

**Example:** In our rolling dice example, the event that the outcome is both even *and* divisible by 3 is $A \cap B = \{6\}$. Therefore,$\mathbf{Pr}[A \cap B] = \frac{1}{6}$.

### Independence

Two events $A$ and $B$ are independent events iff (if and only if) $\mathbf{Pr}[A|B] = \mathbf{Pr}[A]$, which also implies:

$$\mathbf{Pr}[A \cap B] = \mathbf{Pr}[A] \cdot \mathbf{Pr}[B]\,.$$

Two random variables $X$ and $X'$ are independent iff for every $x \in \Omega$ and $x' \in \Omega'$, the two events $X = x$ and $X' = x'$ are independent.

Note that in the rolling dice example we have

$$\mathbf{Pr}[A \cap B] = 1/6 = 1/2 \cdot 1/3 = \mathbf{Pr}[A] \cdot \mathbf{Pr}[B]\,,$$

so the events $A$ and $B$ are *independent*.