COMP 585 | **Spring 2024**

Probabilistic Toolkit for Learning and Computing | *Instructor: Maryam Aliakbarpour*

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Lecture 27

1 Computational Hardness of PAC-Learning

So far, we've assumed that we can easily access and come up with any element in a concept class. Generally, for PAC-learning, we need to find a randomized-polynomial time algorithm.

In this lecture, we show that the concept class of 3-disjunctive normal functions (3-DNF) is not PAC-learnable by proving that there does not exist a randomized-polynomial time algorithm that learns this class.

Definition 1.1 (3-DNF). *We define the concept class, $\mathcal{C}$ of 3-DNFs as:*

$$\mathcal{C} = \{T_1 \vee T_2 \vee T_3 \mid T_i \in \mathcal{T}\}$$

Where $\mathcal{T}$ is the class of conjunctions over $\mathcal{X} = \{x_1, \bar{x_1}, ..., x_m, \bar{x_m}\}$ such that $S_i \subseteq \mathcal{X}$ and:

$$T_i = \bigwedge_{y_j \in S_i} y_j$$

To show the hardness of 3-DNF, we will create a reduction from an NP-hard problem, 3-coloring on a graph.

Definition 1.2 (3-Coloring on a Graph). *Let $G = (V, E)$ be a graph, with $V = [n]$. Then G is 3-colorable if we can assign each vertex of G one of three colors such that no two vertices are the same color.*

Assume we have a 3-coloring over a graph G. Let $S_G = \{(\vec{x}_1, y_1), ..., (\vec{x}_n, y_n)\}$ be a set of labelled samples constructed from this coloring.

Let D be uniform over S_G, let $\epsilon = \frac{1}{2|S_G|}$ and let $\delta \in (0, 1)$. Assume we have a PAC-learning algorithm, $\mathcal{A}$, for 3-DNF. Then, with probability $1 - \delta$, given enough samples from D, $\mathcal{A}$ outputs $\hat{c}$ such that:

$$\text{err}(\hat{c}) \leq \epsilon$$

We note that if $\mathcal{A}$ succeeds, $\hat{c}$ will actually have error 0:

$$\begin{aligned}
\text{err}(\hat{c}) &= \mathbb{P}_{(\vec{x},y)\sim D}(\hat{c}(\vec{x}) \neq y) \\
&= \frac{1}{|S_G|} \sum_{(\vec{x},y)\sim S_G} \mathbb{1}[\hat{c}(\vec{x}) \neq y] \\
&= \frac{\beta}{|S_G|} \text{ for some } \beta \in \mathbb{N} \\
&\leq \epsilon \ \text{ w.p. } 1-\delta \\
&= \frac{1}{2|S_G|} \\
\implies & \beta = 0 \text{ since } \beta \in \mathbb{N} \\
\implies & \text{err}(\hat{c}) = 0
\end{aligned}$$

Then, we are interested in showing that S_G leads to $\hat{c}$ with $\text{err}(\hat{c}) = 0 \iff G$ is 3-colorable.

1.1 G is 3-colorable $\implies$ S_G leads to $\hat{c}$ with err$(\hat{c}) = 0$

Let $G = ([n], E)$.

Construct a set $S_G^+ = \{(v^{(i)}, y^{(i)})\}_{i=1}^n$ corresponding to the vertices of G as follows:

$$v^{(i)} = (1, 1, ..., 1, \overbrace{0}^{\text{index } i}, 1, ..., 1) \in \{0,1\}^n$$
$$y^{(i)} = 1$$

Construct a set $S_G^- = \{(e^{(i,j)}, y^{(i,j)})\}_{(i,j)\in E}$ corresponding to the edges of G as follows:

$$\forall (i,j) \in E,\ e^{(i,j)} = (1, ..., 1, \overbrace{0}^{\text{index } i}, 1, ..., 1, \overbrace{0}^{\text{index } j}, 1, ..., 1) \in \{0,1\}^n$$
$$y^{(i,j)} = 0$$

Let $S_G = S_G^+ \cap S_G^-$. We will show that G 3-colorable $\implies$ we can find a 3-DNF T such that $\text{err}_{S_G}(T) = 0$

Assume G is 3-colorable. Then, we can find a coloring (R, Y, B) such that R, Y, B form a partition of $[n]$. Using this coloring, we define T_R, T_Y, T_B to be used in our 3-DNF, T:

- $T_R(\vec{x}) = \bigwedge_{i:i\notin R} \vec{x}_i$
- $T_Y(\vec{x}) = \bigwedge_{i:i\notin Y} \vec{x}_i$
- $T_B(\vec{x}) = \bigwedge_{i:i\notin B} \vec{x}_i$

- $T = T_R \lor T_Y \lor T_B$

We then prove that $\text{err}_{S_G}(T) = 0$. To achieve this, we want $T(v^{(i)}) = 1$ for all $v^{(i)}$ and $T(e^{(i,j)}) = 0$.

Consider three cases:

1. $v^{(i)} \in R$.

$$\begin{gathered} \implies x_i \notin T_R \text{ and } v_j^{(i)} = 1\ \forall i \neq j \\ \implies T_R(v^{(i)}) = 1 \\ \implies T(v^{(i)}) = 1 \end{gathered}$$

2. $v^{(i)} \notin R$.

$$\begin{gathered} \implies x_i \in T_R \\ \implies x_i \notin T_Y \text{ or } x_i \notin T_B \\ \implies \text{case 1 applies, considering } Y \text{ or } B \text{ instead of } R. \\ \implies T(v^{(i)}) = 1 \end{gathered}$$

3. $v^{(i)} \in R$ and $(i,j) \in E$.

$$\begin{gathered} \implies j \notin R \implies x_j \in T_R \text{ and } e_j^{(i,j)} = 0 \\ \implies T_R(v^{(i)}) = 0 \end{gathered}$$

   WLOG, assume $v^{(j)} \in B$. By the same argument, $T_B(v^{(i)}) = 0$. Then, we have both $v^{(i)} \notin Y$ and $v^{(j)} \notin Y$. So, $x_i, x_j \in T_Y$ and $e_i^{(i,j)} = e_j^{(i,j)} = 0$. So, $T_Y(v^{(i)}) = 0$.

$$\implies T(e^{(i,j)}) = 0$$

So, T correctly labels all samples in S_G.

1.2 S_α leads to $\hat{c}$ with err$(\hat{c}) = 0 \implies G$ is 3-colorable

Given $G = ([n], E)$ and a concept $T = T_R \lor T_Y \lor T_B$ with error 0, we want to find a valid coloring (R, Y, B).

Because T has error 0 on S_G, we know that T satifies:

$$T(v^{(i)}) = 1, \quad T(e^{(i,j)}) = 0$$

Assign vertex i to R if $T_R(v^{(i)}) = 1$. Perform the same process to construct Y and B.

Then, we will show that (R, Y, B) is a valid coloring.

Proof. Let $(i, j) \in E$. We know $T(e^{(i,j)}) = 0 \implies T_R(e^{(i,j)}) = 0$. For contradiction, assume $v^{(i)} \in R$ and $v^{(j)} \in R$. Then, $T_R(v^{(i)}) = 1 \implies x_i \notin T_R$ and $T_R(v^{(j)}) = 1 \implies x_j \notin T_R$.

$$\implies T_R(e^{(i,j)}) = 1 \text{ since } x_i, x_j \notin T_R \text{ and } e_k^{(i,j)} = 1 \ \forall k \neq i, j$$
Contradiction.

So, $v^{(i)}, v^{(j)}$ cannot both be in R. So, we have a valid coloring. □