- Sub Exponentials

Recall

Is every random variable $subG(k)$ for some large k? Nope.

Let $Z \sim N(0,1)$

Consider Z^2

The MGF of Z^2-1 is (centered)

$$E\left[e^{\lambda(Z^2-1)} \right] =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\lambda(z^2-1)} \cdot e^{-z^2/2} \, dz$$

$$= \frac{e^{-\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}(1-2\lambda)} \, dz$$

$$= \begin{cases} \dfrac{e^{-\lambda}}{\sqrt{1-2\lambda}} \overset{\frac{1}{4} \geq \lambda \geq 0}{\leq} e^{\lambda^2/1-2\lambda} \overset{\frac{1}{4} \geq \lambda \geq 0}{\leq} e^{2\lambda^2} & \lambda < \frac{1}{2} \\ \infty & \lambda \geq \frac{1}{2} \end{cases}$$

↳ unbounded :(

for $|\lambda| \leq \frac{1}{4}$: Z^2 is behaving like a sub-Gaussian

Sub-Exponentials mimic the behavior of Z^2.

Sub Exponential random variables

A random variable X with mean $\mu = \mathbb{E}[X]$ is sub-exponential if there are non-negative parameters (s^2, α) s.t.

$$\mathbb{E}\left[e^{\lambda(X-\mu)}\right] \leq e^{s^2\lambda^2/2} \quad \forall |\lambda| < \frac{1}{\alpha}$$

+ $X \sim \text{subG}(s^2) \Rightarrow X \sim \text{SubE}(s^2, 0)$

+ if X is $\text{subG}(k^2) \Rightarrow X^2 \in \text{SubE}$

Equivalent definition of sub Exponentials

Let X be a r.v. The following properties are equivalent; The parameter $c_i > 0$ appearing below differ from each other by at most an absolute constant factor.

① tail bound $\forall t \geq 0$

$$\Pr[|X| \geq t] \leq 2\exp(-t/c_1)$$

② moment bound

$$\|X\|_{L^p} = \left(E[|X|^p]\right)^{1/p} \leq c_2 p \quad \forall p \geq 1$$

③ Moment generating func. of $|X|$

$$E[e^{\lambda|X|}] \leq e^{c_3 \lambda} \quad 0 \leq \lambda \leq \frac{1}{c_3}$$

④ $\exists k_4$ such that: $E[e^{|X|/c_4}] \leq 2$

⑤ MGF of X itself:

if $\mathbb{E}[X] = 0$

$$\mathbb{E}[e^{\lambda X}] \leq e^{c_5^2 \lambda^2}$$

Lemma If $X \in SubE(s^2, \alpha)$

$$\Pr[X - E[X] \geq t] \leq \begin{cases} e^{-t^2/2v^2} & 0 \leq t \leq v^2/\alpha \\ e^{-t/2\alpha} & t \geq v^2/\alpha \end{cases}$$

The same holds for the tail of $-(X - E[X])$

alternatively

$$\Pr[X - E[X] \geq t] \leq \exp\left(-\min\left\{\frac{t^2}{2v^2}, \frac{t}{2\alpha}\right\}\right)$$

$$\leq \exp\left(-\frac{t^2/2}{v^2 + t\alpha}\right)$$

↳ roughly $\theta(-t)$

proof:

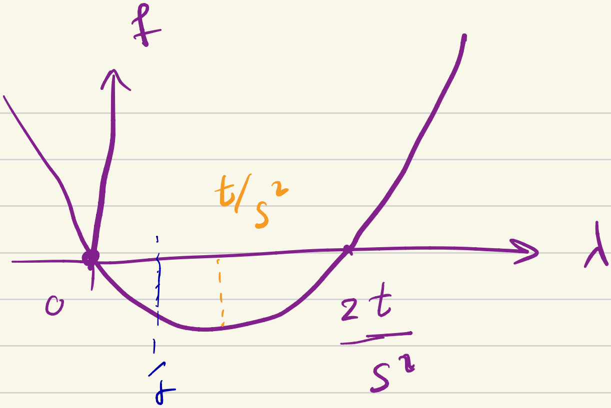
(if $\lambda > 0$)

$$\Pr[X - E[X] \geq t] = \Pr[\lambda(X - E[X]) \geq \lambda t]$$

$$= \Pr[e^{\lambda(X - E[X])} \leq e^{\lambda t}]$$

$$\leq \frac{E[e^{\lambda(X - E[X])}]}{e^{\lambda t}} \leq e^{s^2\lambda^2/2 - \lambda t} \quad *$$

(markov) ($\lambda < \frac{1}{\alpha}$)



$$f(\lambda) = \frac{s^2\lambda^2}{2} - \lambda t$$

if $\quad \frac{t}{s^2} < \frac{1}{\alpha} \quad \Rightarrow \quad \bigstar \leq e^{-\frac{t^2/s^2}{2}}$

if $\quad \frac{1}{\alpha} \leq \frac{t}{s^2} \qquad \bigstar \leq e^{\frac{s^2}{2\alpha^2} - \frac{t}{\alpha}}$

$$\frac{s^2}{2\alpha^2} = \frac{s^2}{2\alpha} \cdot \frac{1}{\alpha} \leq \frac{s^2}{2\alpha} \cdot \frac{t}{s^2} \leq \frac{t}{2\alpha}$$

$$\frac{s^2}{2\alpha^2} - \frac{t}{\alpha} \leq \frac{t}{2\alpha} - \frac{t}{\alpha} \leq -\frac{t}{2\alpha}$$

$$\Rightarrow \bigstar \leq e^{-t/2\alpha}$$