Lecture 1 Jan 9

probability review

property testing

testing sortedness

discrete probability space (Ω, P)

$\Omega = \{\omega_1, \omega_2, \ldots\}$ elementary outcomes.

sample space,

(finite or countable)

event: any subset of Ω

P : events $\longrightarrow [0,1]$

usually defined just by determining $p(\{\omega_i\})$

or $p(\omega_i)$

s.t:

* $\forall$ event $E \subseteq \Omega$

$$p(E) = \sum_{\omega \in E} p(\omega)$$

* $p(\Omega) = 1$

random variable $\rightarrow$ takes value in Ω
based on a probability distribution.

$X \sim P$

* Expected Value

$$E[X] = \sum_{x \in \Omega} p(x) \cdot x$$

$$\int_{x \in \Omega} p(x) \cdot x \, dx$$

* Variance $= E[(X - E[X])^2]$

$= E[X^2] - E[X]^2$

* linearity of expectation:

$$E[X_1 + X_2] = E[X_1] + E[X_2]$$

$$E[\alpha X] = \alpha E[X]$$

$$\Rightarrow Var[\alpha X] = \alpha^2 Var[X]$$

$$Var[X+Y] \neq Var[X] + Var[Y]$$

For two events $A, B \subseteq \Omega$:

* joint probability $Pr[A \cap B]$

probability that both A and B happen.

* Conditional probability $Pr[A \mid B]$

probability that A happens conditioned on that B happens

$$Pr[A \mid B] = \frac{P(A \cap B)}{P(B)}$$

Bayes' Theorem $P(A \mid B) = \dfrac{P(B \mid A) \cdot P(A)}{P(B)}$

* Independence

A and B are independent events

$$\iff \quad P(A \mid B) = P(A)$$

$$\iff \quad Pr(A \cap B) = P(A) . P(B)$$

Two random variable $X \in \Omega$ and $X' \in \Omega'$ are independent iff

$\forall \ x \in \Omega, \ x' \in \Omega'$ two events

$X = x$ and $X' = x'$ are independent.

Randomness in computation

One example of employing randomness to obtain faster algorithms is in designing sub-linear algorithm.

our example today:

Testing sortedness of an array

A	2	8	15	...	102
	1	2	3		n

<u>Def.</u> A is sorted $\iff A[1] \leq A[2] \leq \cdots \leq A[n]$

Any deterministic algorithm needs to query $\Omega(n)$ cells in A to test sortedness.

Can we test sortedness with $o(n)$ queries?

Even a randomized algorithm would require $\Omega(n)$ queries to A.

what is the difficulty? perfectionism

Randomness

- cannot find needle in a haystack
  - ↳ there must be substantial evidence of unsortedness.
- unlikely events may occur against all odds.
  - ↳ It is ok to fail with small prob.

Property testing

property $\mathcal{P}$ = a set of objects

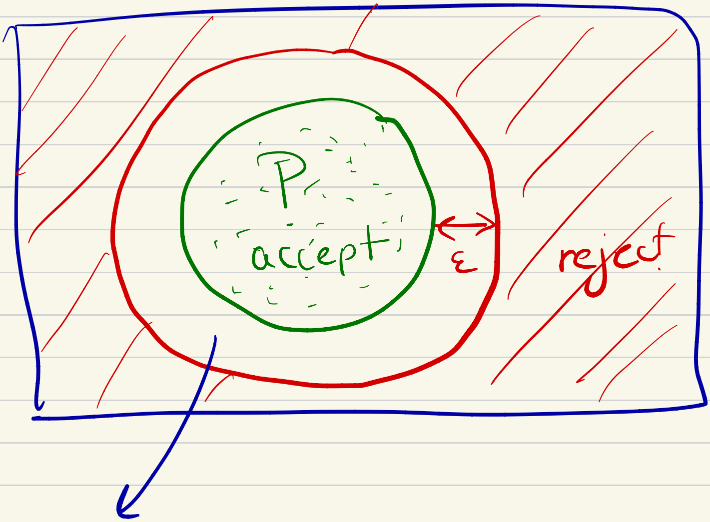
We say an object has property $\mathcal{P}$ iff the object is in $\mathcal{P}$.

Suppose we have an underlying object O

Def We say an algorithm $\mathcal{A}$ is an (ε, δ) - tester for property $\mathcal{P}$ if the following holds with prob. at least $1-\delta$:

if $O \in \mathcal{P}$, $\mathcal{A}$ outputs accept

if O is ε-far from $\mathcal{P}$, $\mathcal{A}$ outputs reject



Both answers are correct.

* ε → make the difference detectable

* δ → leaves room for error whe we get unlucky.

"ε-far" will be defined in the context of the problem.

For sortedness problem, suppose we have two arrays of length n: A and A'

- We say A and A' are ε-far iff one can change A in $\geq \varepsilon \cdot n$ entries to obtain A'.

- We say A is ε-far from being sorted if A is ε-far from all sorted arrays.

Easy case: 0-1 array

proposed Algorithm $\mathcal{A}$:

- Draw m samples. uniformly at random from $[n]$
  $\hookrightarrow \{1, 2, \ldots, n\}$

- sort the samples: $i_1 \le i_2 \le \cdots \le i_m$

If $A[i_1] \le A[i_2] \le \cdots \le A[i_m]$

output accept

Else

output reject.

We aim to find m such that

$\mathcal{A}$ is an (ε, δ) tester.

$$\Pr[\text{wrong answer}] < \delta$$

step 1) if A is sorted, for every $i<j$

we have $A[i] < A[j]$, and there

is no violation. Thus it is impossible

for $\mathcal{A}$ to find one.

$$\Rightarrow \Pr[\text{ outputting reject} \mid \text{sorted } A] = 0$$

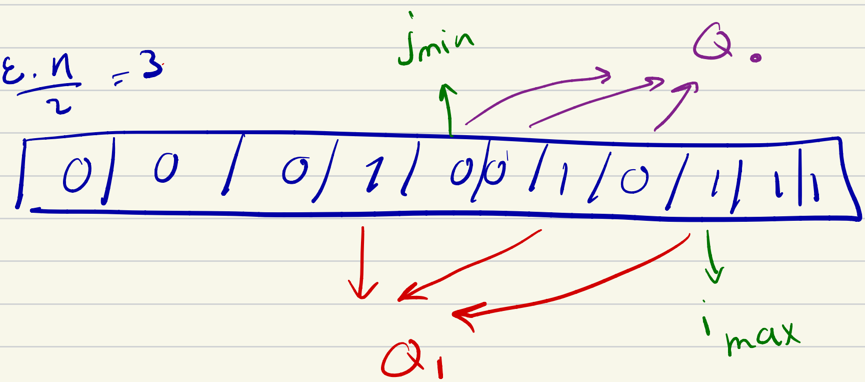
step 2) Next, we show that

$$\Pr\left[\text{Outputting accept} \,\middle|\, \begin{array}{l} A \text{ is } \varepsilon\text{-far} \\ \text{from being sorted} \end{array}\right] < \delta$$

$Q_1 = \{$ set of indices of the left most 1's $\}$

$Q_0 = \{$ set of indices of the right most 0's $\}$

Example if $\frac{\varepsilon \cdot n}{2} = 3$



Lemma 1

if A is ε-far from being sorted $\Rightarrow$

all indices in Q_1 are smaller than all indices in Q_0

(

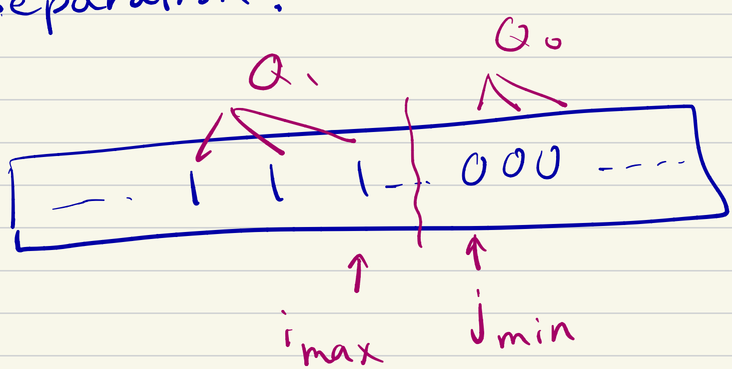
that is $i_{max} < j_{min}$

where j_{min} = min index in Q_0

& i_{max} = max index in Q_1

Pictorially, the lemma says:

if A is ε-far there is a separation:



Proof of Lemma 1:

By contradiction assume:

$$j_{min} < i_{max}$$

we show A cannot be ε-far from being sorted by constructing a sorted A' that is ε-close to A.

$$A'[i] = \begin{cases} 0 & i \leq i_{max} \\ 1 & i > i_{max} \end{cases}$$

Obviously A' is sorted.

distance between A & A'

We change two groups in A to get A'

1) $A[i] = 1$ if $i \leq i_{max}$

2) $A[i] = 0$ if $i > i_{max}$

1) There are exactly $\frac{\varepsilon . n}{2}$ 1's in A that appear before i_{max}

This holds by definition of Q_1.

2) There are less than $\frac{\varepsilon . n}{2}$ 0's that appear after i_{max}

since $j_{min} < i_{max}$

$\Rightarrow$ with $< \varepsilon \cdot n$ changes to A

we get to A', a sorted

array

$\Rightarrow$ A is not ε-far $\cancel{X}$

Hence, $j_{min} > i_{max}$

(they cannot be equal either) $\square$

Suppose A is ε-far

$\Longrightarrow$ indices in Q_1 $<$ indices in Q_0

Lemma 1

if algorithm $\mathcal{A}$ samples

$i \in Q_1$ and $j \in Q_0$

$\Longrightarrow$ $\mathcal{A}$ output reject

why? $A[i] = 1$, $A[j] = 0$

but $i < j$

Define two events

E_1 = at least one sampled index $\in Q_1$

E_0 = 〃 〃 〃 〃 〃 $\in Q_0$

We just showed

$$E_1 \wedge E_2 \Rightarrow \text{outputting reject}$$

Now, let's go back to bounding the probability of wrong answer:

$$\Pr[\text{outputting accept} \mid A \text{ is } \varepsilon\text{-far}]$$

$$\leq \Pr[\overline{E_1 \wedge E_2} \mid A \text{ is } \varepsilon\text{-far}]$$

$$\leq \Pr[\bar{E}_1 \vee \bar{E}_2 \mid A \text{ is } \varepsilon\text{-far}]$$

$$\leq \Pr[\bar{E}_1] + \Pr[\bar{E}_2] \qquad \text{union bound}$$

$$\leq 2\Pr[\bar{E}_1] \qquad \text{symmetry}$$

We pick m random index in $[n]$

$$\Pr[\text{one sample} \in Q_1] = \frac{|Q_1|}{n} = \frac{\varepsilon}{2}$$

$$\Pr[\text{one sample} \notin Q_1] = \frac{|Q_1|}{n} = 1 - \frac{\varepsilon}{2}$$

$$\Pr[E_1] = \Pr[\text{one sample among } m \text{ samples} \notin Q_1]$$

$$= \left(1 - \varepsilon/2\right)^m$$

$$= e^{\varepsilon/2 \cdot m}$$

using $1 - x < e^{-x}$

$$\leq \frac{\delta}{2}$$

set $m = \frac{2}{\varepsilon} \log \frac{2}{\delta}$

$\Rightarrow$ $\Pr[\text{outputting accept} \mid A \text{ being } \varepsilon\text{-far}] < \delta$

$\Rightarrow$ Hence, A with $m = \frac{2}{\varepsilon} \log 2/\delta$ is an (ε, δ)-tester for sortedness.