Homework 6

Instruction for NP-Completeness Proof

In this assignment, you will be asked to prove the NP-completeness of several problems. As discussed in class, follow these steps to demonstrate that a problem B is NP-complete:

1. **Choose a known NP-complete problem *A*.** Your goal is to provide a polynomial-time reduction $A \leq_p B$.

2. **Define the reduction.** For any instance x of A, describe how to construct a corresponding instance y of B. Additionally, assuming a black-box algorithm solves instance y, describe how to use that result to determine the solution for x.

   Alongside your explanation, provide pseudocode for this reduction and briefly argue that it runs in polynomial time.

3. **Prove correctness.** Establish the biconditional relationship between the instances of A and B:

   $$x \in A \iff y \in B$$

   - ($\Rightarrow$) If x is a YES instance of A, demonstrate how a solution (certificate) for x yields a valid solution for y.
   - ($\Leftarrow$) If y is a YES instance of B, demonstrate how a valid solution for y yields a valid solution for x.

4. **Conclude NP-completeness.** First, verify that B is in NP. Since you have already shown that B is NP-hard (via $A \leq_p B$), and now that $B \in$ NP, you may conclude that B is NP-complete.

Problem 1. (30 points) Undirected Hamiltonian Cycle

A *Hamiltonian cycle* in a graph is a cycle that visits every vertex exactly once. In class, we established that determining whether a **directed** graph $G = (V, E)$ contains a Hamiltonian cycle is NP-complete. In this problem, we demonstrate that the **undirected** version is NP-complete as well by providing a polynomial-time reduction from the directed Hamiltonian cycle problem to the undirected one.

Hint: A similar idea to vertex splitting might be useful for this problem.

Problem 2. (35 points) Strongly Independent Set

A *Strongly Independent Set* in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that for every distinct pair $u, v \in S$, the distance between them is strictly greater than 2 (i.e., $d(u, v) \geq 3$). Prove that determining whether a graph G contains a Strongly Independent Set of size at least k is NP-complete. You should provide a polynomial-time reduction from the standard **Maximum Independent Set (MIS)** problem. For simplicity, assume that there is no degree zero vertex in your graphs. Also, assume $k \geq 2$, as the problem is trivial otherwise.

Hint: Consider subdividing every edge in the original graph to increase distances. How can you then modify the graph to ensure none of the new "subdivision" vertices are selected in an optimal solution?

Problem 3. (35 points) Path-Selection

Suppose you are managing a communication network, modeled by an undirected graph $G = (V, E)$. There are c users who want to use the network. User $i \in [c]$ issues a *request* to reserve a specific path P_i in G on which to transmit data. You want to accept as many path requests as possible, except if you accept P_i and P_j, then P_i and P_j cannot share any nodes. The Path-Selection problem formalizes this intent. It asks: Given an undirected graph $G = (V, E)$, a set of path requests $P_1, \ldots, P_c$, and a number k, is it possible to select at least k of the paths so that no two of the selected paths share any nodes? Show that Path-Selection is NP-complete.