COMP 382: Reasoning about Algorithms

P, NP, NP-Hardness, NP-Completeness

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Today's Lecture

1. **What Is NP-Hardness?**

Reading:

- Chapter 19 of [Roughgarden, 2022]

Content adapted from the same reference.

What Is NP-Hardness?

The Core Problem: Selection Bias

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- They focus on problems with clever, fast algorithms (e.g., sorting, shortest paths, MSTs).
- Many important problems have **no fast algorithms known**.
- These problems are deemed "intractable."

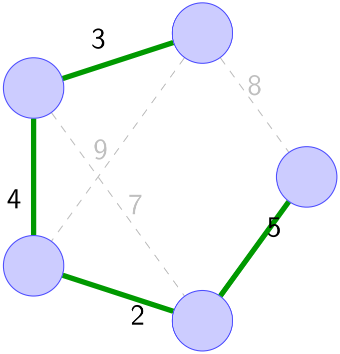
MST vs TSP

An Algorithmic Mystery

“Easy”: Minimum Spanning Tree (MST)

Problem: Find a spanning tree (a subset of edges that connects all vertices without cycles) of minimum total edge cost.

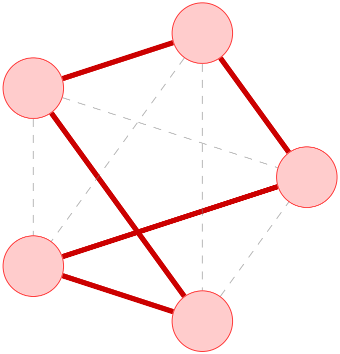
- Solvable by blazingly fast algorithms:
  - Prim’s
  - Kruskal’s
- **Running Time:** $O((m+n)\log n)$.
- This is a **computationally easy** problem.



"Hard": Traveling Salesman Problem (TSP)

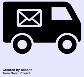
Problem: Find a tour (a cycle visiting every vertex exactly once) of minimum total edge cost.

- The definition looks deceptively similar to MST.
- No fast algorithm is known.
- Exhaustive search is $O(n!)$, which is **infeasible**.
- This is **computationally hard**.



Why TSP Matters: Real-World Intractability

TSP is a powerful template for many practical optimization problems.



Mail Deliveries

finding the shortest route for deliveries.

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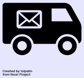

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Genome Sequencing

Finding the most plausible ordering of overlapping gene fragments.

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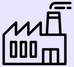


Factory Assembly

Minimizing setup costs between assembling different car models.

Defining “Easy” and “Hard” Problems

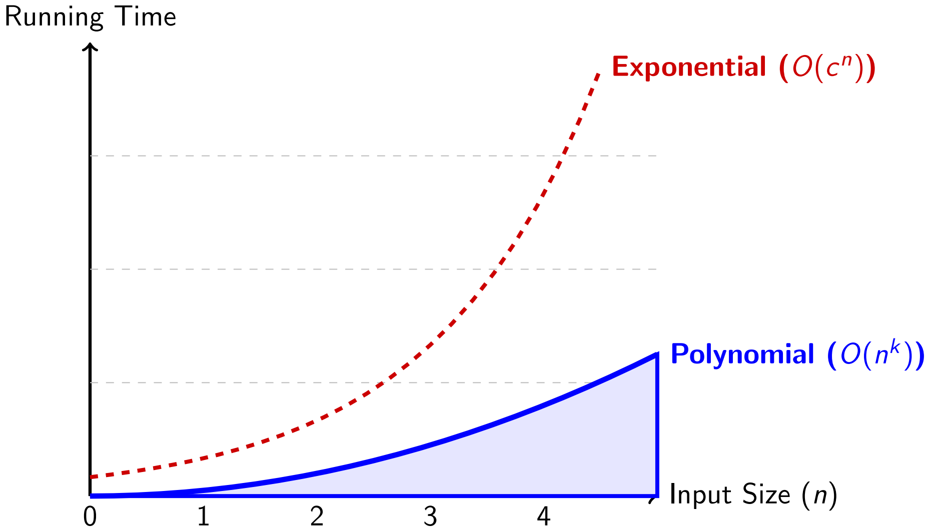
Or, a gentle introduction to complexity classes

Easy and Hard Problems

An oversimplified view:

- **Easy:** can be solved with a polynomial-time algorithm.
- **Hard:** require exponential time in the worst case.

Polynomial vs. Exponential Time

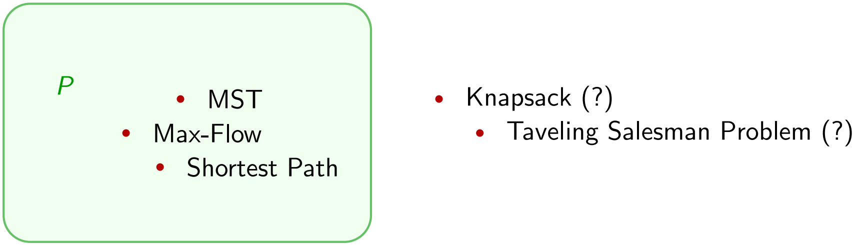


P: Polynomial Time Solvable Problems

- Complexity theory classifies problems based on their *inherent difficulty*;
- Algorithms can be fast or slow, clever or naive, but our statements about the *problem itself*.
- A problem is polynomial time solvable if there is an algorithm that correctly solves it in $O(n^k)$ time, for some constant k, where n is the input length.
- still polynomial even $k = 10^{10}$.
- This is worst-case running time. (maximum running time over all possible inputs of size n)
- **P**: Problems solvable in **P**olynomial time (easy to **solve**).

P: Examples

- Typical examples:
  - Shortest paths (without nasty conditions like negative cycles).
  - Minimum spanning tree, maximum flow, bipartite matching, etc.
- Non-example: the standard dynamic programming for knapsack runs in $\Theta(nW)$ time, where W is the capacity; since the input size is only $\log W$, this is actually **pseudopolynomial**, not polynomial, in the input length.



Decision Problems: The Formal Foundation

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Decision

- **MST (Decision):** Is there a spanning tree with total cost $\leq k$?
- **TSP (Decision):** Is there a tour with total cost $\leq k$?

Optimization

- **MST (Optimization):** Find the minimum cost spanning tree.
- **TSP (Optimization):** Find the shortest tour.

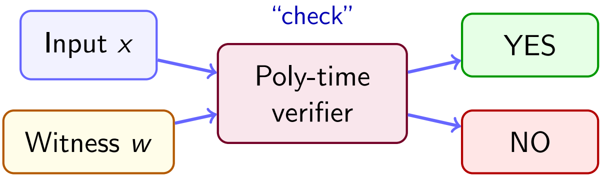
The Class NP

The Class NP

NP is the class of problems for which *solutions can be efficiently recognized*, even if we don't know how to find them efficiently.

A problem is in NP if:

- YES-instances have short **witnesses** (certificates) whose length is polynomial in the input size.
- We can verify a witness in polynomial time.



Decision Version of TSP and Its Witness

- **Input:** Complete graph $G = (V, E)$ with edge lengths d_{uv} and a budget k.
- **Question:** Is there a tour (Hamiltonian cycle) of total length $\leq k$?

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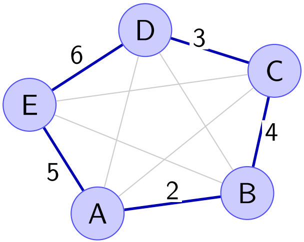
$$\text{witness: } A \to B \to C \to D \to E \to A\,, \qquad k = 25\,.$$



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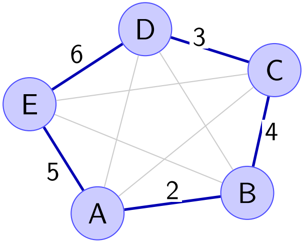

tour 1: $2 + 4 + 3 + 6 + 5 = 20$ Yes!

Solving TSP via Brute-Force Algorithm

1. Enumerate all possible tours (Hamiltonian cycles) on V.
2. For each tour C:
   - Check it visits every vertex exactly once.
   - Compute its total length $L(C)$.
   - If $L(C) \leq k$, **accept**.
3. If no tour passes the test, **reject**.

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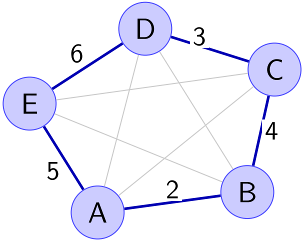

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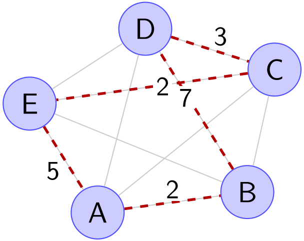

tour 2: $A \to B \to D \to C \to E \to A$

$2 + 7 + 3 + 2 + 5 = 19$ (better)

Solving TSP via Brute-Force Algorithm

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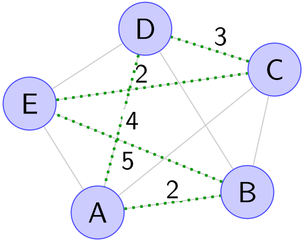
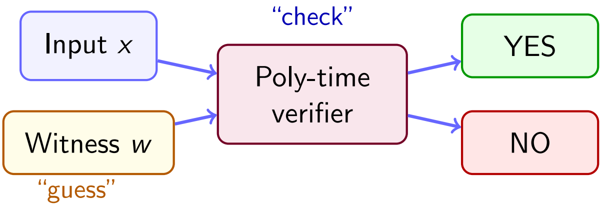


tour 3: $A \to B \to E \to C \to D \to A$

$2 + 5 + 2 + 3 + 5 = 17$ (best)

The Class NP as "Guess and Check"

- For problems in NP, we can always solve them by:
  1. Enumerating all candidate solutions (witnesses) of polynomial length. [guess a solution]
  2. Checking each one using the polynomial-time verifier.
- Number of candidates is typically exponential in input size $\Rightarrow$ exponential-time brute force.
- Vast majority of important natural problems (scheduling, routing, puzzles, many optimization problems) live in NP.



What Does “NP” Stand For?

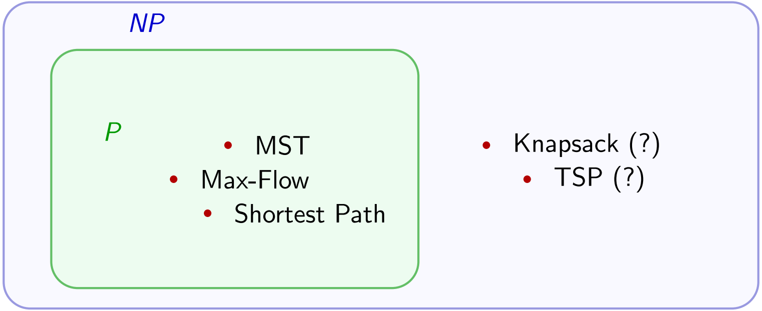
- Common wrong guess: “not polynomial”.
- Correct: **Nondeterministic Polynomial time**.

- Historically defined using *nondeterministic Turing machines*: machines that can “guess” a solution and then verify it in polynomial time.
- Modern viewpoint (equivalent and more intuitive for us): NP is the set of problems with polynomial-time verifiers and polynomial-length witnesses.

Is P $=$ NP?

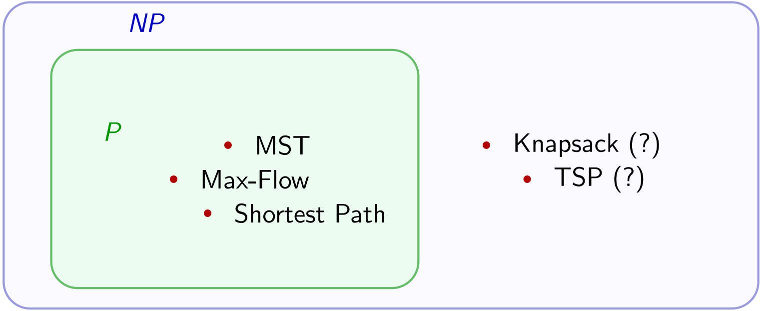
Is P = NP?

- We know that $\mathbf{P} \subseteq \mathbf{NP}$, e.g., MST $\in$ NP.



Is P = NP?

- We know that **P** $\subseteq$ **NP**, e.g., MST $\in$ NP.
- For many problems in NP, no polynomial-time algorithm is known, (e.g., TSP).



The P vs. NP Conjecture

Conjecture: $\mathbf{P} \neq \mathbf{NP}$. Most experts believe this is true.

> *If P=NP, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in "creative leaps," no fundamental gap between solving a problem and recognizing the solution once it's found. Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss; everyone who could recognize a good investment strategy would be Warren Buffett. It's possible to put the point in Darwinian terms: if this is the sort of universe we inhabited, why wouldn't we already have evolved to take advantage of it?*

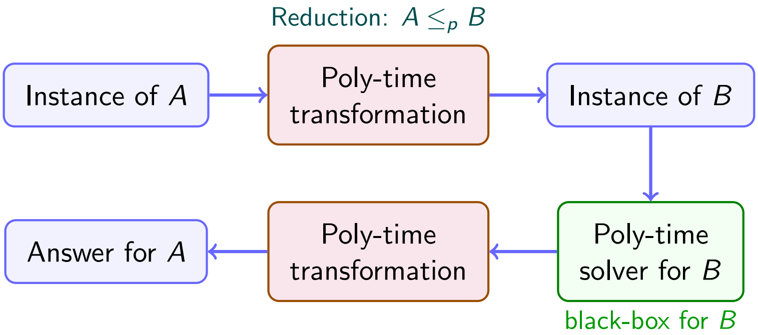
— Scott Aaronson, on Shtetl-Optimized

Reductions

Comparing Problem Difficulty

Reductions as Black-Box Transformations

- To show problem A is **no harder** than B:



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- To show problem A is **no harder** than B:
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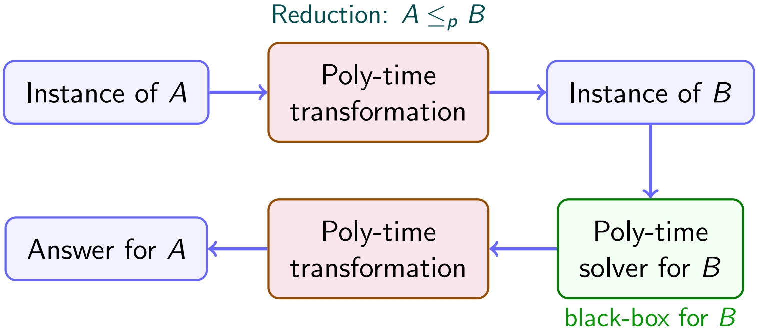

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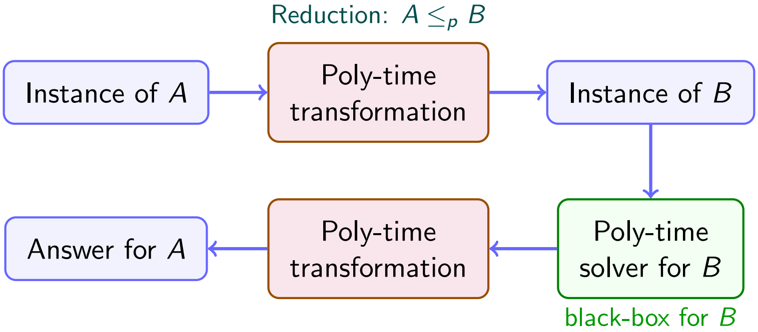
  - Use a black-box solver for B.



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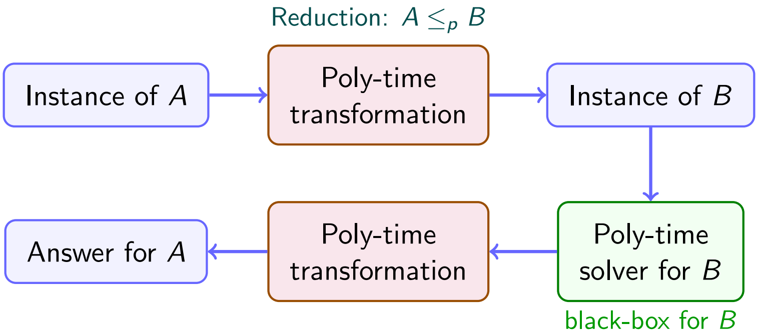

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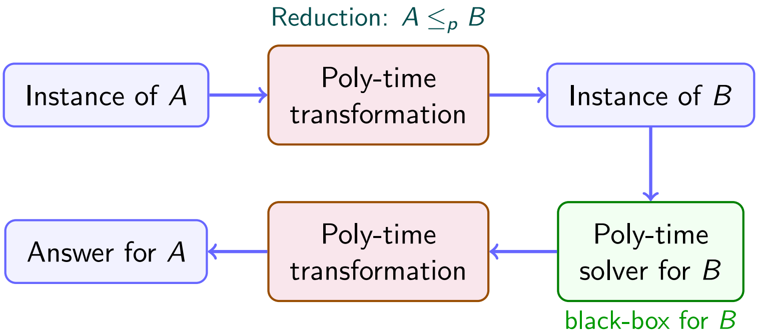
- To show problem A is **no harder** than B:
  - Convert any instance of A to an instance of B in polynomial time.
  - Use a black-box solver for B.
  - Convert the answer back to an answer for A.
- If B is easy (in P), then A is also easy.



Reductions: Comparing Problem Difficulty

Big idea: If B were easy (poly-time), then A would also be easy.

Problem A **reduces** to problem B if, given a polynomial-time subroutine ("oracle") for B, we can solve A in polynomial time.

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- We'll use reductions to show many problems are "as hard as" TSP.
- Examples:
  - Computing the median reduces to sorting.
  - Detecting a cycle in a graph reduces to depth-first search.
  - All-pairs shortest paths reduces to repeated single-source shortest paths.

NP-hardness and NP-Completeness

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- If you find a polynomial-time algorithm for an NP-hard problem, then *every* problem in NP becomes easy (poly-time).
- That is B is as hard as any problem in NP, or it could be even harder.

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- B is one of the "hardest" problems in NP.

Example: TSP is an NP-complete problem.

Is TSP as Hard as All Problems?

No! There are problems that are not even *computable*.

The Halting Problem:

Input: a program & an input.
Question: will the program eventually halt on that input?

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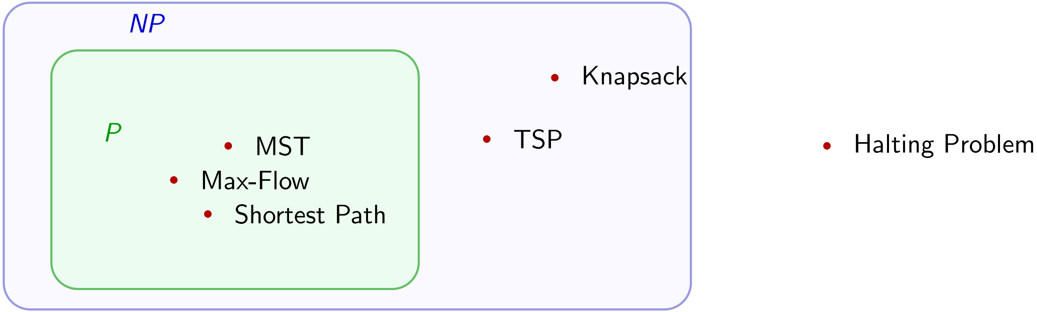
The halting problem is NP-hard.

The Landscape



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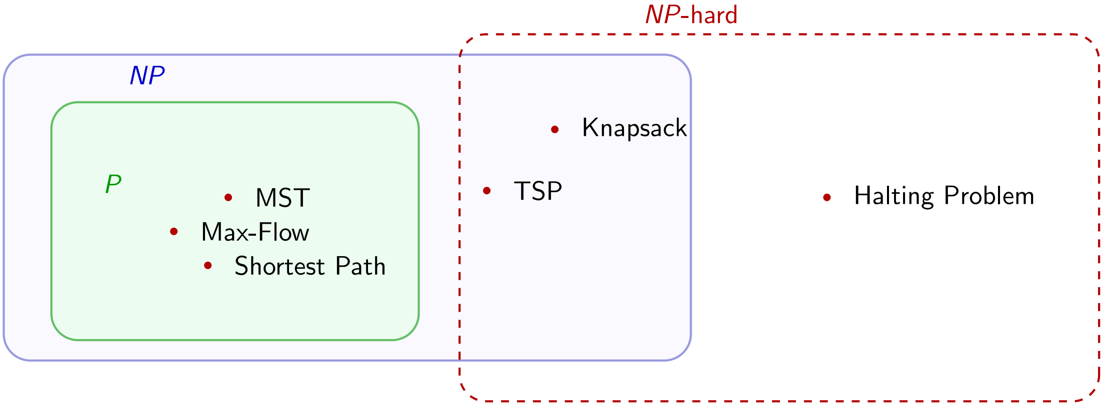

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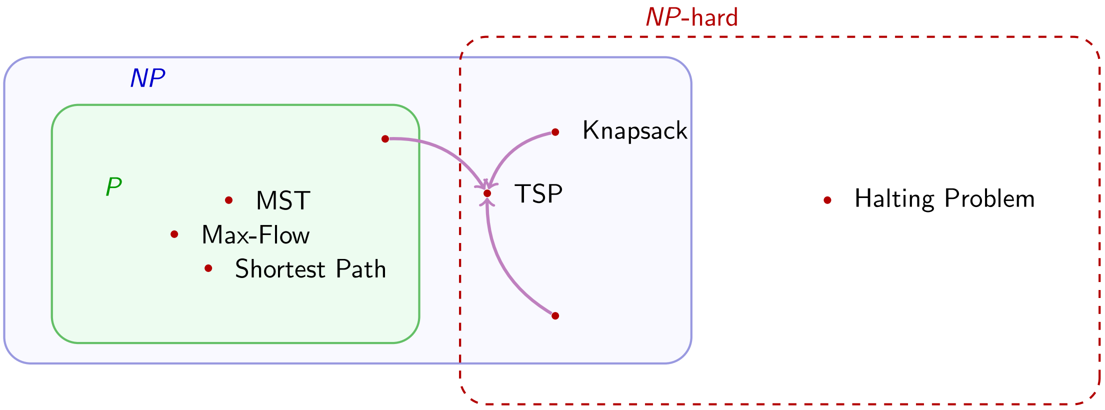

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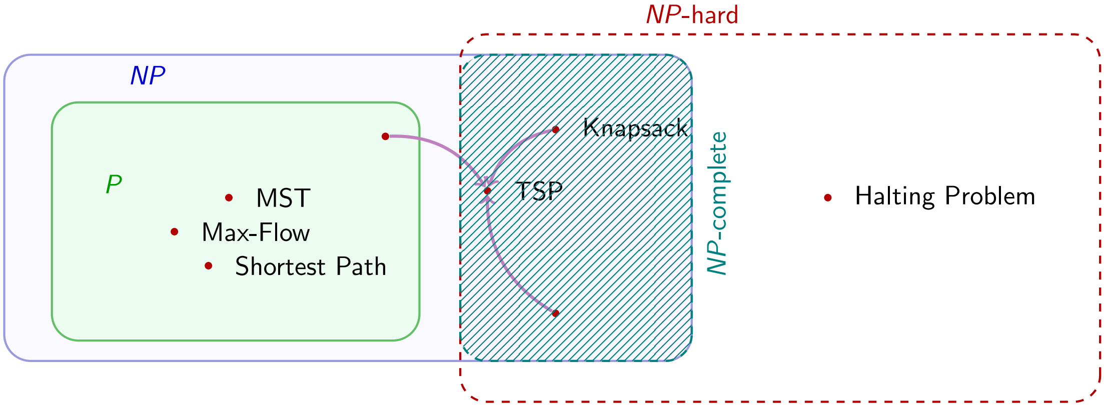

So your problem is NP-complete... now what?

- NP-complete does *not* mean "hopeless."
- It means: no known polytime algorithm for *all* inputs.
- We change strategy:

1. Special cases (easy structure)
2. Heuristics & approximations
3. Smarter exponential-time algorithms

Circuit Satisfiability (CIRCUIT-SAT)

where it all began

The Strange Power of NP-Completeness

It is a very strange concept.

- How can we argue that *every* problem in NP reduces to one particular problem?
- Is one problem really complex enough to capture all the nuances of every problem in NP?
- Did someone actually sit down and write a reduction from *all* NP problems to a single one?
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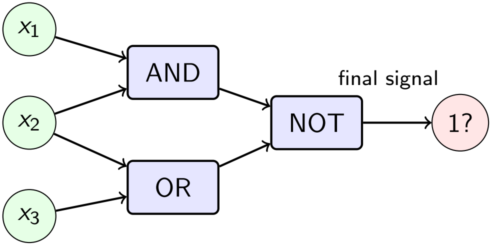
This was the breakthrough of Cook–Levin: they proved that CIRCUIT-SAT is powerful enough to express *any* NP computation.

Circuit Satisfiability (CIRCUIT-SAT)

Input: A Boolean circuit C with input bits $x_1, \ldots, x_n$ (built from AND, OR, NOT gates).

Question: Is there an assignment to $(x_1, \ldots, x_n)$ such that the output of C is $\mathbf{1}$?

- **Interpretation:** Think of the circuit as a little machine of logic gates. We ask whether there exists an input vector that makes the output wire "turn on".
- CIRCUIT-SAT is the **first NP-complete** problem (Cook and Levin 1971).



CIRCUIT-SAT Captures All of NP

Cook–Levin Theorem (1971): CIRCUIT-SAT is NP-complete.

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   Proof: Next!

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For every decision problem $A \in \mathrm{NP}$, each input instance x has a definite answer:

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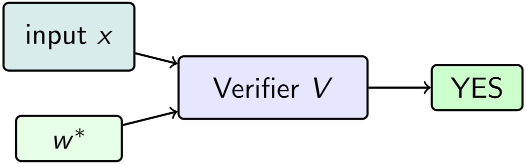
What characterizes problems in NP is how their **YES-instances** behave:

- If x is a **YES-instance** of A, then there exists a polynomial-size **witness** w that certifies this.
- There is a polynomial-time **verifier** $V(x, w)$ that checks whether w is a valid witness for x.

What NP Really Means

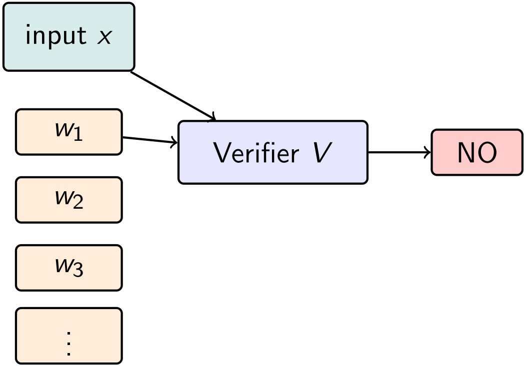
For YES-instances:

$\exists w$ such that $V(x, w) = \text{YES}$.



For NO-instances:

$$\forall w,\ V(x, w) = \text{NO}.$$



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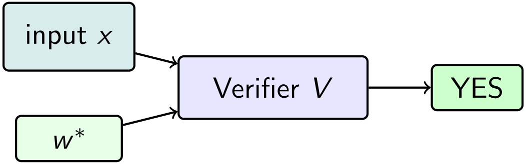
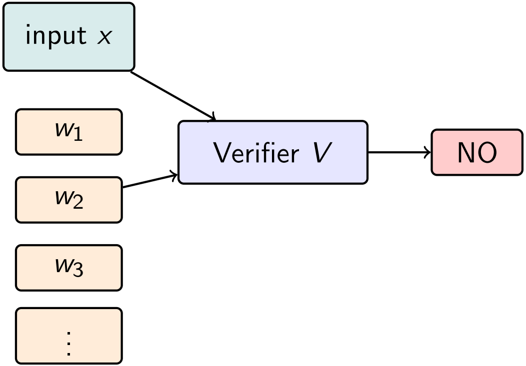
For YES-instances:

$\exists w$ such that $V(x, w) = \text{YES}$.



For NO-instances:

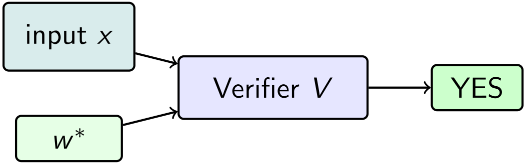
$$\forall w,\ V(x, w) = \text{NO}.$$



What NP Really Means

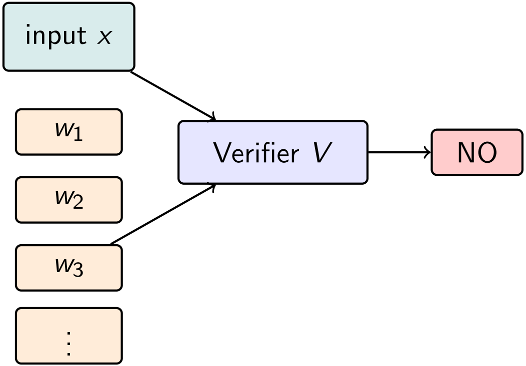
For YES-instances:

$\exists w$ such that $V(x, w) = \text{YES}$.



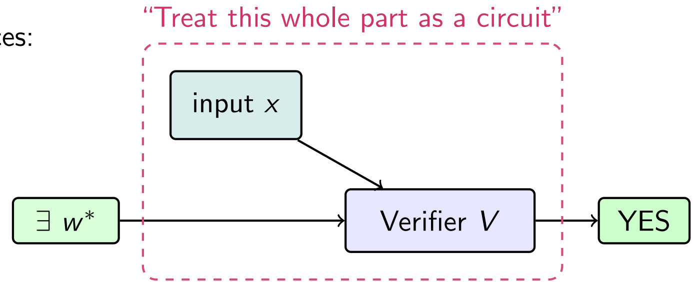
For NO-instances:

$$\forall w,\ V(x, w) = \text{NO}.$$

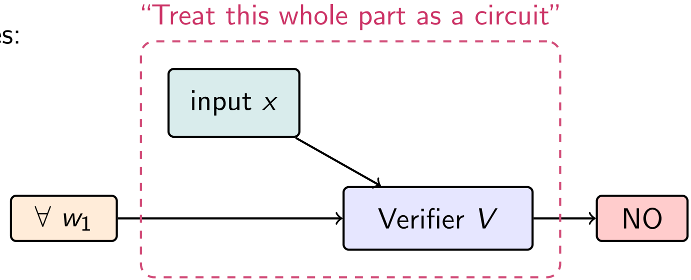


NP literally means CIRCUIT-SAT

For YES-instances:



For NO-instances:



Cook–Levin Theorem

Key idea: SAT (via CIRCUIT-SAT) can act as a *universal witness finder* for every problem in NP.

For any decision problem $A \in \mathrm{NP}$ and any instance x of A:

- If x is a **YES**-instance, then there exists a witness w that convinces the verifier $V(x, w)$ to accept.
- If x is a **NO**-instance, then *no* witness can make the verifier accept.

The Cook–Levin theorem encodes this verifier behavior into a CIRCUIT-SAT formula. Given an instance x of A, we construct a Boolean formula Φ_x such that:

$$\Phi_x \text{ is satisfiable} \quad \Longleftrightarrow \quad \exists w : V(x, w) = \text{accept}.$$

Thus SAT simulates the entire accepting computation of the verifier— it captures the witness *and* every step showing that the witness is correct.

References

Roughgarden, T. (2022).
Algorithms Illuminated: Omnibus Edition.
Soundlikeyourself Publishing, LLC.