COMP 382: Reasoning about Algorithms

Linear Programming & Duality

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Today's Lecture

1. **Linear Programming**

2. **What Is NP-Hardness?**

Reading:

- Lecture note in [Goemans, 2015]
- Lecture note in [Trevisan, 2011]
- Chapter 19 of [Roughgarden, 2022]

Content adapted from the same references.

Linear Programming

Problems with Linear Constraints

- Making the best choice under limits (budget, time, capacity).
- When relationships are *linear*, we get **Linear Programming (LP)**.
- LP appears in scheduling, transport, game theory, and machine learning.

Next: real-life examples

The Diet Problem

- We must plan a daily diet using two grains: G_1 and G_2.
- Each grain provides *carb, protein, and vitamins*, and has a cost per kg.
- Goal: meet daily nutritional requirements **at minimum cost**.

	Carb	Protein	Vitamins	Cost ($/oz)
G_1	5	4	2	0.60
G_2	7	2	1	0.35

Requirements per day: 8 units carb, 15 units protein, 3 units vitamins.

The Diet Problem

Variables (amount/day): $x_1 \leftarrow$ amount of G_1, $x_2 \leftarrow$ amount of G_2

$$\min \ 0.6x_1 + 0.35x_2$$

$$\begin{aligned} 5x_1 + 7x_2 &\geq 8 && \text{(carb)} \\ 4x_1 + 2x_2 &\geq 15 && \text{(protein)} \\ 2x_1 + x_2 &\geq 3 && \text{(vitamins)} \\ x_1, x_2 &\geq 0 \end{aligned}$$

Interpretation: pick amounts to meet each need as cheaply as possible.

The Transportation Problem

Two factories F_1, F_2 and three cities C_1, C_2, C_3.

	C_1	C_2	C_3	Supply
F_1	5	5	3	6
F_2	6	4	1	9
Demand	8	5	2	

Minimize total cost subject to all supplies and demands being met.

The Transportation Problem

Decision variables: x_{ij} = thousands of widgets shipped from F_i to C_j.

$$
\begin{aligned}
\min 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + x_{23} & \\
x_{11} + x_{21} = 8 & \qquad \text{(demand } C_1\text{)} \\
x_{12} + x_{22} = 5 & \qquad \text{(demand } C_2\text{)} \\
x_{13} + x_{23} = 2 & \qquad \text{(demand } C_3\text{)} \\
x_{11} + x_{12} + x_{13} = 6 & \qquad \text{(supply } F_1\text{)} \\
x_{21} + x_{22} + x_{23} = 9 & \qquad \text{(supply } F_2\text{)} \\
x_{ij} \geq 0 & \qquad \text{(no negative shipments)}
\end{aligned}
$$

Interpretation: ship goods to meet all demands at minimum total cost.

What is Linear Programming?

Definition

A **linear program (LP)** optimizes a linear function subject to a set of linear equality or inequality constraints.

- We can always rewrite any LP in a **canonical form**.
- Geometry: intersection of half-spaces (a polyhedron).
- Algorithms: solved efficiently (e.g., *Simplex method*).

From real problems to canonical form

Linear programs can look very different:

$$\min 2x_1 - x_2 \quad \text{s.t.} \quad \begin{cases} x_1 + x_2 \geq 2, \\ 3x_1 + 2x_2 \leq 4, \\ x_1 + 2x_2 = 3, \\ x_1 \text{ free}, \ x_2 \geq 0. \end{cases}$$

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To solve any LP systematically or design algorithms for them, we need to convert it into a unified template...

Canonical Form

$$\max \; c^\top x \quad \text{s.t. } Ax \leq b, \; x \geq 0$$

- x: decision variables
- c: objective coefficients
- A: constraint matrix, b: resource limits

Every LP can be written in this form by adding slack variables or sign changes.

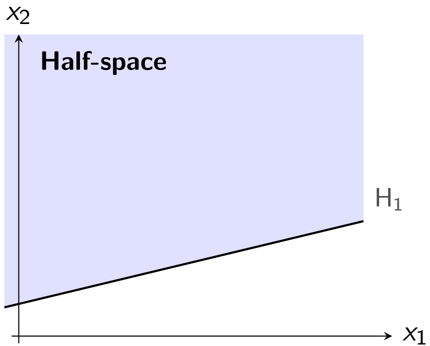
Feasibility Region: From half-spaces to polygons

Step 1. Half-space.

One inequality defines a line and the side that satisfies it.

$$\frac{x_1}{3} - x_2 \leq -1$$

Feasible set: *half-space*.

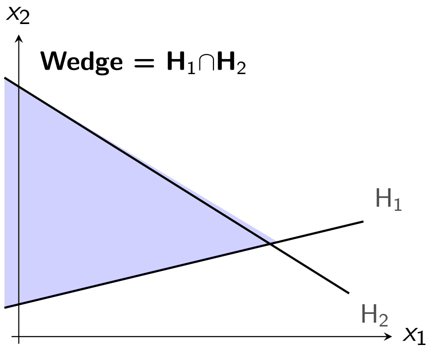


Feasibility Region: From half-spaces to polygons

Step 2. Wedge.

Two inequalities ⇒ intersection of two half-spaces.

Feasible set: *wedge* (two half-spaces).

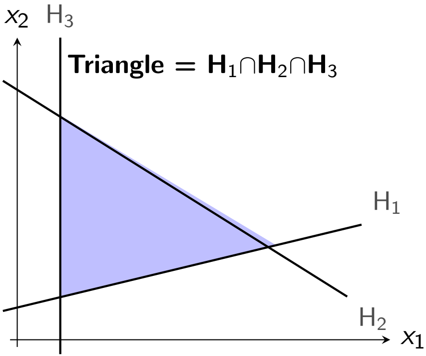


Feasibility Region: From half-spaces to polygons

Step 3. Triangle.

A third inequality can bound the region in 2D.

Feasible set: *triangle* (bounded).

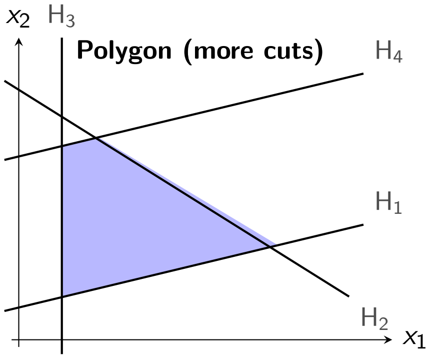


Feasibility Region: From half-spaces to polygons

Step 4. Polygon.

Additional constraints cut off corners
⇒ refined feasible set.

Feasible set: *polygon*.

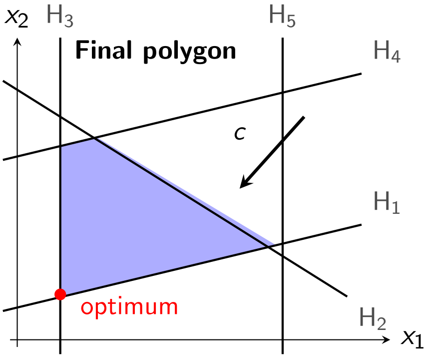


Feasibility Region: From half-spaces to polygons

Step 5. Optimum at a vertex.

Maximizing $c^\top x$ pushes along c to (usually) a vertex of the polygon.

Feasible set: *polygon*;



Simplex Method

A short overview

Simplex Method

- Every LP's feasible region is a **polyhedron** (a polygon in 2D, polytope in 3D).

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- Each move improves the objective value — and there are finitely many vertices.

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  1. Starts from one feasible vertex.
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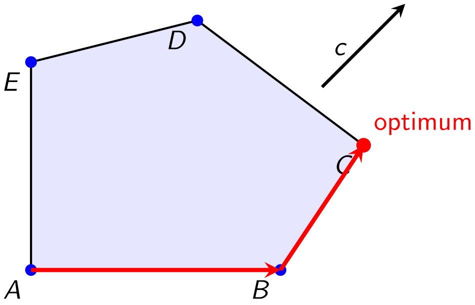
  3. Stops when no further improvement is possible.
- Each move improves the objective value — and there are finitely many vertices.
- Simplex always ends at an **optimal vertex** (if one exists).

Simplex Path on a Polygon (2D intuition)

Each step: move along an edge to a better vertex.

"Walk around the polygon" until no edge improves the objective.



Time Complexity of the Simplex Method

- $n \leftarrow$ number of variables

- In the **worst case**, there can be exponentially many vertices:

$$\text{Worst case: } O(2^n)$$

  (Klee–Minty cube example).

- In **practice**, Simplex is extremely fast — polynomial time.

- Theoretical guarantee (polynomial time) comes from **interior-point methods**

Duality in Linear Programming

An Example of Duality

Primal:

$$\max z = 5x_1 + 4x_2$$

$$\text{s.t.} \begin{cases} x_1 \leq 4 & (1) \\ x_1 + 2x_2 \leq 10 & (2) \\ 3x_1 + 2x_2 \leq 16 & (3) \\ x_1, x_2 \geq 0 \end{cases}$$

- Feasible solution $(x_1, x_2) = (4, 2)$ gives $z = 28 \implies$ lower bound.
- Multiply (3) by 2: $6x_1 + 4x_2 \leq 32 \implies z \leq 32 \implies$ upper bound.
- Adding (1)+(2)+(3): $5x_1 + 4x_2 \leq 30 \implies z \leq 30$.

Combining Inequalities to Bound the Optimum

Multiply constraints by nonnegative multipliers y_1, y_2, y_3:

$$(y_1 + y_2 + 3y_3)x_1 + (2y_2 + 2y_3)x_2 \leq 4y_1 + 10y_2 + 16y_3.$$

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To ensure an upper bound on $z = 5x_1 + 4x_2$, impose:

$$y_1 + y_2 + 3y_3 \geq 5, \quad 2y_2 + 2y_3 \geq 4.$$

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To ensure an upper bound on $z = 5x_1 + 4x_2$, impose:

$$y_1 + y_2 + 3y_3 \ge 5, \quad 2y_2 + 2y_3 \ge 4.$$

Then minimize the RHS $4y_1 + 10y_2 + 16y_3$.

Dual:

$$\min\ w = 4y_1 + 10y_2 + 16y_3$$
$$\text{s.t.} \quad \begin{cases} y_1 + y_2 + 3y_3 \ge 5, \\ 2y_2 + 2y_3 \ge 4, \\ y_1, y_2, y_3 \ge 0. \end{cases}$$

Verifying Optimality via Duality

- We have established that for any pair of feasible solutions:

$$z(x) \leq w(y)$$

- Try $(x_1, x_2) = (3, 3.5) \implies z = 5(3) + 4(3.5) = 29.$
- Try $(y_1, y_2, y_3) = (0, 0.5, 1.5) \implies w = 4(0) + 10(0.5) + 16(1.5) = 29.$

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- Therefore, when they match, **both are optimal:** $z^* = w^* = 29.$

Duality provides **certificates of optimality**: when a feasible x and y give equal objective values, they must be optimal.

Duality in Canonical Form

$$\text{(P)} \;\; \max c^\top x \quad \text{s.t.} \;\; Ax \le b, \;\; x \ge 0$$
$$\text{(D)} \;\; \min b^\top y \quad \text{s.t.} \;\; A^\top y \ge c, \;\; y \ge 0$$

- Each primal constraint $\Rightarrow$ dual variable.
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- The two problems are mirrors of one another.

Weak Duality

$$c^\top x \leq y^\top Ax \leq y^\top b$$

- For any feasible x (primal) and y (dual): $z = c^\top x \leq w = b^\top y$.
- Dual feasible solutions give *upper bounds* on the primal optimum.

Convention: $\max \emptyset = -\infty$, $\min \emptyset = +\infty \implies$ always $z^* \leq w^*$.

Strong Duality

If both (P) and (D) have feasible solutions and one is bounded, then both attain the same finite optimum.

$$z^* = w^*$$

- Proof idea: simplex optimality conditions produce a dual feasible y with equal objective value.

Summary of primal–dual relationships

	Dual finite	Dual unbounded	Dual infeasible
Primal finite	$z^* = w^*$	impossible	impossible
Primal unbounded	impossible	impossible	possible
Primal infeasible	impossible	possible	possible

Interpretation:

- If one is unbounded, the other is infeasible.
- If one has a finite optimum, so does the other, with equal value.
- Both can be infeasible simultaneously.

Max-Flow Min-Cut Theorem with LP Duality

Max-Flow as a Linear Program

Given a directed network $(G = (V, E), s, t, c)$ with capacities $c(u, v)$:

- We can formulate max-flow problem as an LP over variables $f(u, v)$ for each edge $(u, v) \in E$.
- Optimal value = value of the maximum s–t flow.
- Assuming there are no incoming edges to s and no outgoing edges from t.

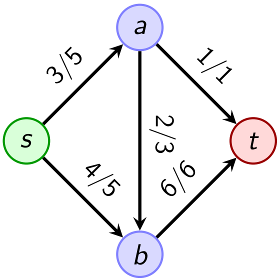
$$\max \quad \sum_{v:(s,v)\in E} f(s, v)$$

$$\text{s.t.} \quad \sum_{u:(u,v)\in E} f(u, v) = \sum_{w:(v,w)\in E} f(v, w), \quad \forall v \in V \setminus \{s, t\} \quad \text{(flow conservation)}$$

$$0 \le f(u, v) \le c(u, v), \qquad \forall (u, v) \in E \quad \text{(capacity)}$$

Flow Decomposition into Paths

By the flow decomposition theorem, max-flow can be viewed as set of s-t paths.



Total Flow f = 7

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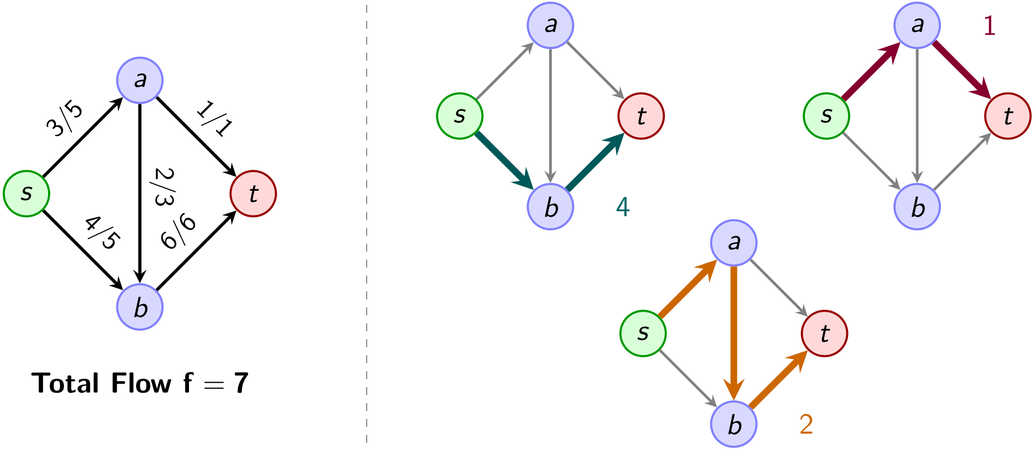

Alternative view: Path-Based LP Formulation

- Let $\mathcal{P}$ be the set of all simple s–t paths, and for each path $p \in \mathcal{P}$, let x_p be the amount of flow sent along p (possibly exponentially many).

$$\begin{aligned}
\max \quad & \sum_{p\in\mathcal{P}} x_p \\
\text{s.t.} \quad & \sum_{p\in\mathcal{P}:(u,v)\in p} x_p \le c(u,v), \quad \forall (u,v)\in E \quad \text{(capacity)} \\
& x_p \ge 0, \quad \forall p \in \mathcal{P}.
\end{aligned}$$

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$\max_f \; |f| = \mathrm{OPT}_{\text{primal}}$

Next: Very clean dual!

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Next: Very clean dual!

Dual of the Path-Based LP

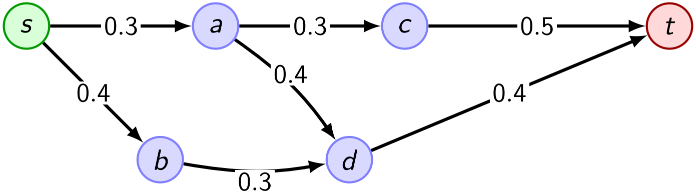
Dual variables: $y_{u,v} \geq 0$ for each edge $(u, v) \in E$.

Dual LP:

$$
\begin{aligned}
\min \quad & \sum_{(u,v)\in E} c(u,v)\, y_{u,v} \\
\text{s.t.} \quad & \sum_{(u,v)\in p} y_{u,v} \;\geq\; 1, && \forall s\text{–}t \text{ paths } p \in \mathcal{P} \\
& y_{u,v} \;\geq\; 0, && \forall (u,v) \in E.
\end{aligned}
$$

Interpretation of Dual

- Interpret $y_{u,v}$ as a length on edge (u, v).



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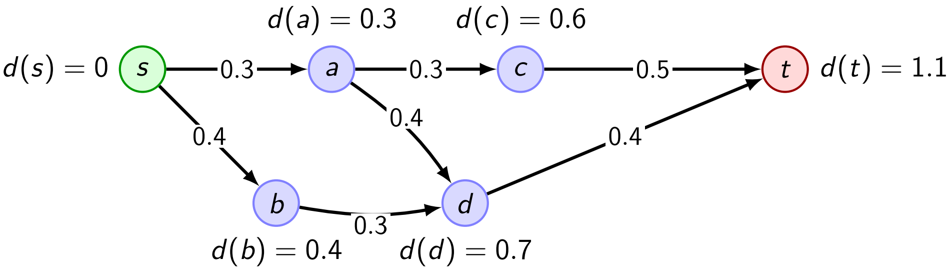
  $\Rightarrow$ in the metric defined by y, distance$(s, t) \geq 1$.



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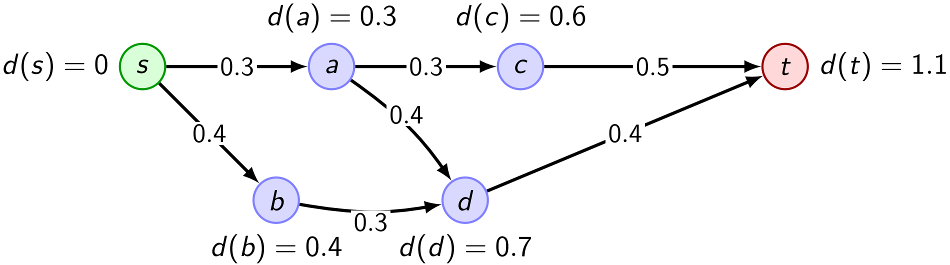
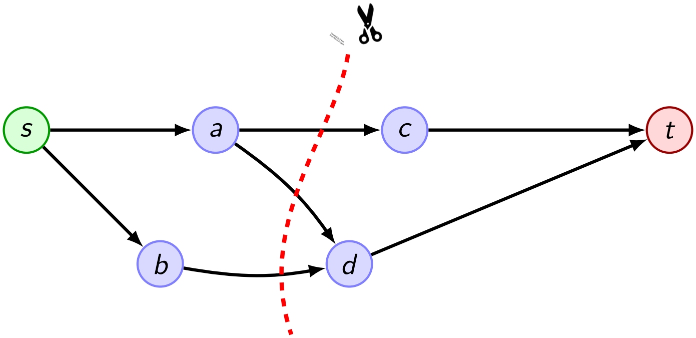
- Interpret $y_{u,v}$ as a length on edge (u, v).
- Constraint: every s–t path has total length at least 1.
  $\Rightarrow$ in the metric defined by y, distance$(s, t) \geq 1$.
- Objective: minimize the capacity-weighted sum of edge lengths.



Cuts $\Rightarrow$ Feasible Dual Solutions

- Given an $(s-t)$-cut A, define

$$y_{u,v} := \begin{cases} 1 & \text{if } u \in A,\ v \notin A \text{ (edge crosses the cut)}, \\ 0 & \text{otherwise.} \end{cases}$$



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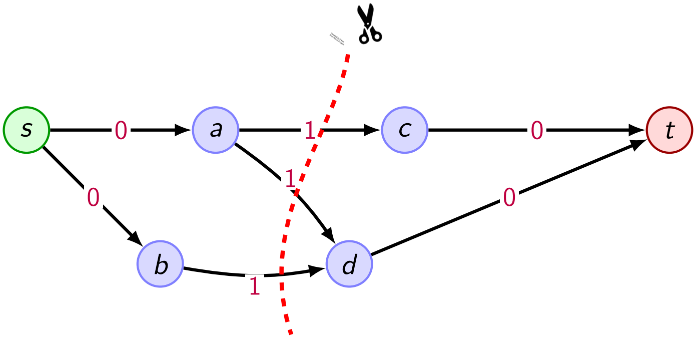

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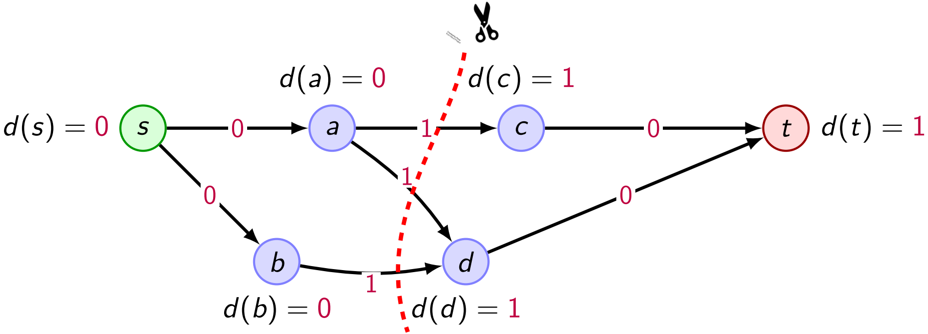


Cuts $\Rightarrow$ Feasible Dual Solutions

- Every s–t path must cross the cut at least once, so the path constraints hold:

$$\sum_{(u,v)\in p} y_{u,v} \geq 1\,.$$

- Dual objective value:

$$\mathrm{OPT}_{\mathrm{dual}} \leq \sum_{(u,v)\in E} c(u,v)\, y_{u,v} \;=\; \sum_{u\in A,\, v\notin A} c(u,v) \;=\; \mathrm{capacity}(A)\,.$$

- Therefore,

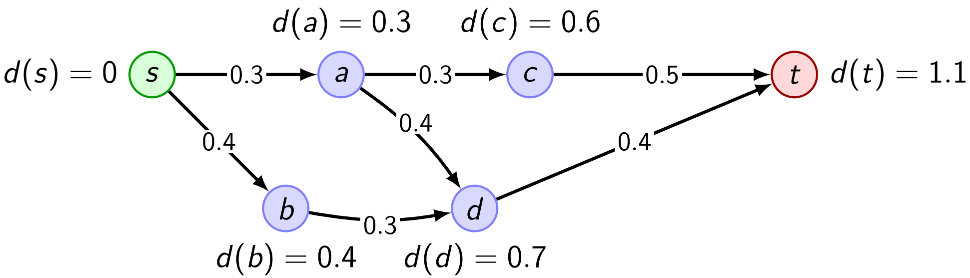
$$\boxed{\mathrm{OPT}_{\mathrm{dual}} \;\leq\; \min_{(s\text{–}t)\ \mathrm{cuts}\ A} \mathrm{capacity}(A)\,.}$$

Dual $\Rightarrow$ Cut

Now go in the other direction: from any dual solution y to a cut.

Step 1: Distances from s

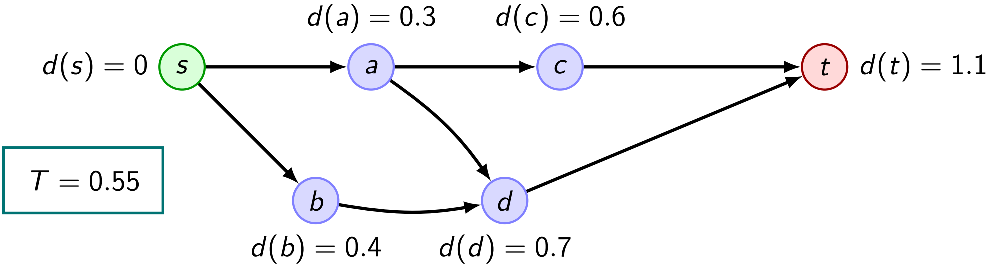
- Compute $d(v)$ = shortest-path distance from s to v (e.g., Dijkstra).
- Dual constraints $\Rightarrow d(t) \geq 1$.



Randomized Rounding: Dual $\Rightarrow$ Cut

Step 2: Random threshold

- Pick T uniformly at random in $[0, 1)$.

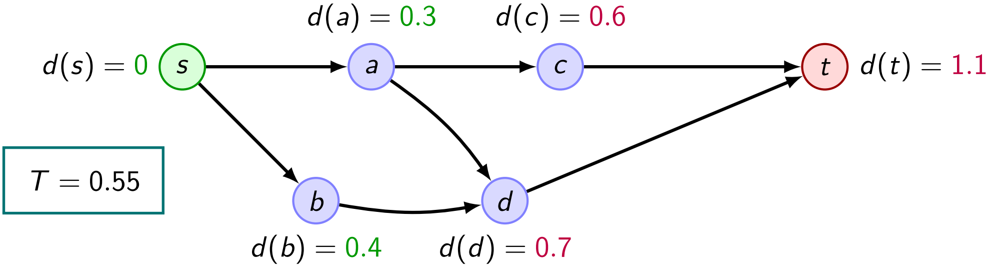


Randomized Rounding: Dual $\Rightarrow$ Cut

Step 2: Random threshold

- Pick T uniformly at random in $[0, 1)$.
- Define the random cut

$$A := \{v \in V : d(v) \leq T\}.$$



Randomized Rounding: Dual ⇒ Cut

Step 2: Random threshold

- Pick T uniformly at random in $[0, 1)$.
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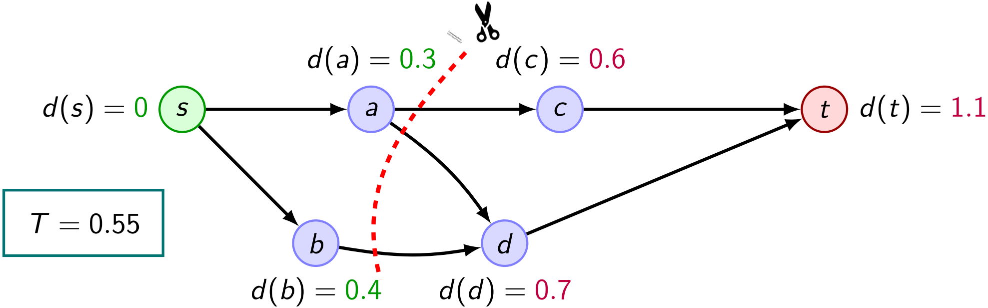
- Then $s \in A$ but $t \notin A$, so A is always a valid s–t cut.



Probability of Being a Cut Edge

For an edge (u, v), what is the probability of $u \in A$, and $v \notin A$?

- If $d(u) > d(v) \implies u$ and v will not be part of a cut.

Probability of Being a Cut Edge

For an edge (u, v), what is the probability of $u \in A$, and $v \notin A$?

- If $d(u) > d(v) \implies u$ and v will not be part of a cut.
- So assume $d(u) \leq d(v)$:

$$\mathbf{Pr}[u \in A,\ v \notin A] = \mathbf{Pr}[d(u) \leq T < d(v)] \leq d(v) - d(u)$$

provided $0 \leq d(u) \leq d(v) \leq 1$ (other cases only make this smaller).

Probability of Being a Cut Edge

For an edge (u, v), what is the probability of $u \in A$, and $v \notin A$?

- If $d(u) > d(v) \implies u$ and v will not be part of a cut.
- So assume $d(u) \leq d(v)$:

$$\mathbf{Pr}[u \in A,\ v \notin A] = \mathbf{Pr}[d(u) \leq T < d(v)] \leq d(v) - d(u)$$

  provided $0 \leq d(u) \leq d(v) \leq 1$ (other cases only make this smaller).
- Shortest-path distances satisfy

$$d(v) \ \leq \ d(u) + y_{u,v}, \implies d(v) - d(u) \leq y_{u,v}$$

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Bounding the Expected Capacity

Given any dual solution y, expected capacity:

$$\begin{aligned}\mathbf{E}_T[\text{capacity}(A)] &= \sum_{(u,v)\in E} c(u,v)\,\mathbf{Pr}[u \in A,\ v \notin A]. \\ &\leq \sum_{(u,v)\in E} c(u,v)\,y_{u,v}\,.\end{aligned}$$

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$$\leq \sum_{(u,v)\in E} c(u,v)\, y_{u,v}\,.$$

- Averaging principle: There exists a (deterministic) choice of T^* with:

$$\text{capacity}(A_{T^*}) \leq \sum_{(u,v)\in E} c(u,v)\, y_{u,v}\,.$$

- Hence,

$$\min_{(s-t)\ \text{cuts}\ A} \text{capacity}(A) \ \leq\ \text{OPT}_{\text{dual}}\,.$$

Dual $\Leftrightarrow$ Min-Cut

We have shown:

- Any cut A gives a feasible dual solution:

$$\mathrm{OPT}_{\mathrm{dual}} \;\leq\; \min_{(s\text{–}t)\text{ cuts } A} \mathrm{capacity}(A)\,.$$

- Given any dual solution y, we can round it to a cut:

$$\min_{(s\text{–}t)\text{ cuts } A} \mathrm{capacity}(A) \;\leq\; \mathrm{OPT}_{\mathrm{dual}}\,.$$

Combining:

$$\min_{(s\text{–}t)\text{ cuts } A} \mathrm{capacity}(A) = \mathrm{OPT}_{\mathrm{dual}}\,.$$

LP Duality $\Rightarrow$ Max-Flow = Min-Cut

- We have shown:

$$\max_f \; |f| = \mathrm{OPT}_{\mathrm{primal}}$$

$$\min_{(s-t) \text{ cuts } A} \mathrm{capacity}(A) = \mathrm{OPT}_{\mathrm{dual}} \,.$$

- Strong Duality implies:

$$\mathrm{OPT}_{\mathrm{primal}} = \mathrm{OPT}_{\mathrm{dual}}$$

- Putting all of these together implies

$$\max_f \; |f| = \min_{(s-t) \text{ cuts } A} \mathrm{capacity}(A)$$

What Is NP-Hardness?

The Core Problem: Selection Bias

- Introductory algorithm books suffer from **selection bias**.

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- They focus on problems with clever, fast algorithms (e.g., sorting, shortest paths, MSTs).
- Many important problems have **no fast algorithms known**.
- These problems are deemed “intractable.”

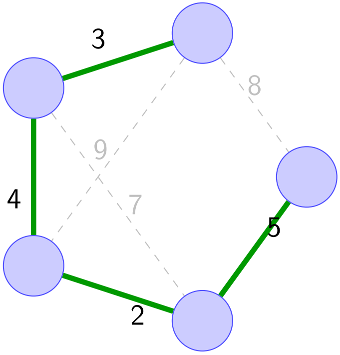
MST vs TSP

An Algorithmic Mystery

"Easy": Minimum Spanning Tree (MST)

Problem: Find a spanning tree (a subset of edges that connects all vertices without cycles) of minimum total edge cost.

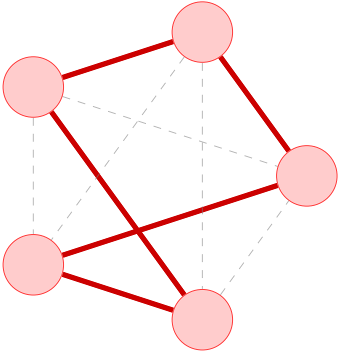
- Solvable by blazingly fast algorithms:
  - Prim's
  - Kruskal's
- **Running Time:** $O((m + n) \log n)$.
- This is a **computationally easy** problem.



"Hard": Traveling Salesman Problem (TSP)

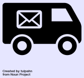
Problem: Find a tour (a cycle visiting every vertex exactly once) of minimum total edge cost.

- The definition looks deceptively similar to MST.
- No fast algorithm is known.
- Exhaustive search is $O(n!)$, which is **infeasible**.
- This is **computationally hard**.



Why TSP Matters: Real-World Intractability

TSP is a powerful template for many practical optimization problems.



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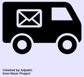

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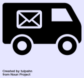

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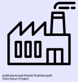


Factory Assembly

Minimizing setup costs between assembling different car models.

Defining “Easy” and “Hard” Problems

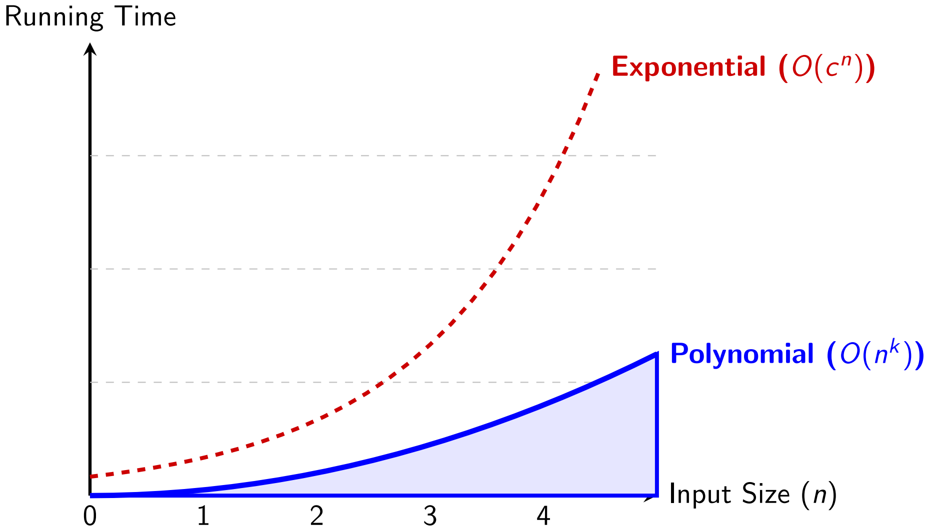
Or, a gentle introduction to complexity classes

Easy and Hard Problems

An oversimplified view:

- **Easy:** can be solved with a polynomial-time algorithm.
- **Hard:** require exponential time in the worst case.

Polynomial vs. Exponential Time



P: Polynomial Time Solvable Problems

- Complexity theory classifies problems based on their *inherent difficulty*;
- Algorithms can be fast or slow, clever or naive, but our statements about the *problem itself*.
- A problem is polynomial time solvable if there is an algorithm that correctly solves it in $O(n^k)$ time, for some constant k, where n is the input length.
- still polynomial even $k = 10^10$.
- This is worst-case running time. (maximum running time over all possible inputs of size n)
- **P**: Problems solvable in **P**olynomial time (easy to **solve**).

NP: Nondeterministic Polynomial time

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- For many problems in NP, no polynomial-time algorithm is known, (e.g., TSP).

- A problem is NP-hard if *every* NP problem reduces to it.

Decision Problems: The Formal Foundation

- Complexity classes are formally defined using problems that yield a simple **YES or NO** answer.
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Decision

- **MST (Decision):** Is there a spanning tree with total cost $\leq k$?
- **TSP (Decision):** Is there a tour with total cost $\leq k$?

Optimization

- **MST (Optimization):** Find the minimum cost spanning tree.
- **TSP (Optimization):** Find the shortest tour.

The P vs. NP Conjecture

Conjecture: $\mathbf{P} \neq \mathbf{NP}$. Most experts believe this is true.

> *If P=NP, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in "creative leaps," no fundamental gap between solving a problem and recognizing the solution once it's found. Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss; everyone who could recognize a good investment strategy would be Warren Buffett. It's possible to put the point in Darwinian terms: if this is the sort of universe we inhabited, why wouldn't we already have evolved to take advantage of it?*

— Scott Aaronson, on Shtetl-Optimized

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- A fast algorithm for one NP-hard problem (like TSP) would solve **thousands** of other unsolved problems.
- This powerful implication is the "strong evidence" of its intractability.

Algorithmic Strategies

The "You Can't Have It All" Principle

An algorithm for an NP-hard problem cannot be all three (assuming $\mathbf{P} \neq \mathbf{NP}$):

General-Purpose Solves all possible inputs.

Correct Always finds the optimal solution.

Fast Runs in polynomial time.

You must compromise on at least one.

Three Algorithmic Strategies

- **Compromise on Generality:** Solve only **special cases** or constrained versions of the problem.
  - *Example:* Weighted Independent Set on path graphs is easy, but on general graphs is NP-hard.

- **Compromise on Correctness:** Use **heuristics** (e.g., Greedy, Local Search).
  - They are fast but may not be optimal. Good for "approximate" answers.

- **Compromise on Speed:** Use an **exact algorithm** that is faster than exhaustive search, but still exponential.
  - *Example:* Dynamic Programming for TSP, or sophisticated SAT/MIP Solvers.

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